J. DIFFERENTIAL GEOMETRY
8 (1973) 349–358

ON THE PRODUCT OF SCHUBERT CLASSES

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1. Introduction

1.1. In his paper [1] Kostant has described the generalized Schubert classes which serve as a basis of the cohomology ring of a large class of homogeneous spaces. The problem investigated here is that of determining the product of two Schubert classes as a linear combination of the others. The extensive notation needed to discuss this question is recalled in § 2. In § 3 some preliminary results are developed, and it is shown that it is sufficient to study the case of the generalized flag manifolds. § 4 contains the main result in which it is shown how the application of a certain linear operator to the product of two Schubert classes yields the product in terms of the other classes. § 5 contains some general statements about the products, including formulas applicable in some simple cases.

2. Background

2.1. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, and let $\mathfrak{k} \subset \mathfrak{g}$ be a fixed compact real form of $\mathfrak{g}$. So $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ is a real direct sum; and the Cartan-Killing form, denoted by $(\, , \,)$, is negative definite on $\mathfrak{k}$. This permits a $*$-operation to be defined on $\mathfrak{g}$ by $(x + iy)^* = -x + iy$ for $x, y \in \mathfrak{k}$. For any subspace $\mathfrak{s}$, $\mathfrak{s}^* = \{x^* \mid x \in \mathfrak{s}\}$.

Let $\mathfrak{b} \subset \mathfrak{g}$ be a fixed Borel subalgebra. Then $\mathfrak{h} = \mathfrak{b} \cap \mathfrak{b}^*$ is a Cartan subalgebra. Let $\Delta \subset \mathfrak{h}'$, the dual of $\mathfrak{h}$, be the set of roots associated with $\mathfrak{h}$. If $\mathfrak{m} = \{x \in \mathfrak{g} \mid (x, y) = 0 \ \forall y \in \mathfrak{b}\}$, then $\mathfrak{b} = \mathfrak{h} + \mathfrak{m}$ and $\mathfrak{g} = \mathfrak{b} + \mathfrak{m}^*$. Both $\mathfrak{m}$ and $\mathfrak{m}^*$ are maximal nilpotent subalgebras, and they are both $\mathfrak{h}$-modules under the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. Therefore $\mathfrak{m}$ is the complex span of $\{e_\varphi \mid \varphi \in \Delta(\mathfrak{m})\}$ for a well-defined subset $\Delta(\mathfrak{m}) \subset \Delta$. Similary, $\mathfrak{m}^*$ is the span of $\{e_\varphi \mid \varphi \in \Delta(\mathfrak{m}^*)\}$. One can show that e_φ^* is a nonzero multiple of $e_{-\varphi}$, so that $\Delta(\mathfrak{m}^*) = -\Delta(\mathfrak{m})$; and one can describe a lexicographic ordering in $\mathfrak{h}'$ for which the positive roots $\Delta_+ = \Delta(\mathfrak{m})$ and the negative roots $\Delta_- = \Delta(\mathfrak{m}^*)$. Finally, one can normalize the root vectors $\{e_\varphi \mid \varphi \in \Delta\}$ so that both $(e_\varphi, e_{-\varphi}) = 1$ and $e_\varphi^* = e_{-\varphi}$ are satisfied. This is the normalization we shall assume hereafter. If $x_\varphi \in \mathfrak{h}$ denotes the root normal corresponding to the root φ, then the following product formulas hold:

Communicated by B. Kostant, March, 13, 1972.

$$[e_\varphi, e_{-\varphi}] = x_\varphi, \qquad [x_\varphi, e_\varphi] = (\varphi, \varphi)e_\varphi, \qquad [x_\varphi, e_{-\varphi}] = -(\varphi, \varphi)e_{-\varphi} .$$

2.2. More generally, suppose $\mathfrak{u}$ is any fixed parabolic subalgebra: $\mathfrak{b} \subset \mathfrak{u} \subset \mathfrak{g}$. Put $\mathfrak{g}_1 = \mathfrak{u} \cap \mathfrak{u}^*$. If $\mathfrak{n} = \{x \in \mathfrak{g} \mid (x, y) = 0 \ \forall y \in \mathfrak{u}\}$, then $\mathfrak{u} = \mathfrak{g}_1 + \mathfrak{n}$ and $\mathfrak{g} = \mathfrak{u} + \mathfrak{n}^*$. Both $\mathfrak{n}$ and $\mathfrak{n}^*$ are $\mathfrak{g}_1$-modules under the adjoint action of $\mathfrak{g}_1$ on $\mathfrak{g}$. So $\mathfrak{n}$ is the complex span of $\{e_\varphi \mid \varphi \in \Delta(\mathfrak{n})\}$; similarly for $\mathfrak{n}^*$. From $\mathfrak{b} \subset \mathfrak{u}$, it follows that $\mathfrak{n} \subset \mathfrak{m}$, $\mathfrak{n}^* \subset \mathfrak{m}^*$, and $\mathfrak{h} \subset \mathfrak{g}_1$.

Define the subspace $\mathfrak{r} = \mathfrak{n} + \mathfrak{n}^*$. $\mathfrak{r}$ is not a Lie subalgebra of $\mathfrak{g}$; but we make it a Lie algebra by giving it the product of $\mathfrak{g}$ on $\mathfrak{n}$ and $\mathfrak{n}^*$ and putting $[x, y] = 0$ for $x \in \mathfrak{n}$ and $y \in \mathfrak{n}^*$. Now consider the exterior algebra $\wedge \mathfrak{r}$. Since $\mathfrak{n}$ and $\mathfrak{n}^*$ are $\mathfrak{g}_1$-modules, $\mathfrak{r}$ and $\wedge \mathfrak{r}$ are too. Denote the invariants by C, that is, C is the subspace of $\wedge \mathfrak{r}$ each of whose elements is annihilated by every $x \in \mathfrak{g}_1$.

2.3. We now introduce a series of operators on C, recalling some facts about some of them. These operators will play an important role in the solution of our problem. Let $\partial \in \operatorname{End} \wedge \mathfrak{r}$ be the usual boundary operator, and $b \in \operatorname{End} \wedge \mathfrak{r}$ be the corresponding coboundary operator, that is, the negative transpose of ∂ with respect to the Cartan-Killing form on $\wedge \mathfrak{r}$. Since C is stable under the action of ∂ and b, we denote by these same symbols their restrictions to C. Now denote by $d \in \operatorname{End} \wedge \mathfrak{g}$ the usual coboundary operator on $\wedge \mathfrak{g}$. In general, $\wedge \mathfrak{r}$ is not closed under the action of d, but it turns out that C is, and we denote by this same symbol its restriction to C.

2.4. Consider the vector space C and the linear operators ∂, b, and d. It has been shown that ∂ and d are disjoint, that is, $d\partial x = 0$ implies $\partial x = 0$ and $\partial d y = 0$ implies $dy = 0$ for $x, y \in C$. Because d and ∂ are disjoint, the laplacian operator $S = d\partial + \partial d$ induces a Hodge decomposition of C: $C = \operatorname{Ker} S + \operatorname{Im} S$. In this direct sum decomposition $\operatorname{Ker} S = \operatorname{Ker} \partial \cap \operatorname{Ker} d$ and $\operatorname{Im} S = \operatorname{Im} \partial + \operatorname{Im} d$. Also $\operatorname{Ker} \partial = \operatorname{Ker} S + \operatorname{Im} \partial$ and $\operatorname{Ker} d = \operatorname{Ker} S + \operatorname{Im} d$. So, if $H(C, \partial)$ denotes the homology group $\operatorname{Ker} \partial / \operatorname{Im} \partial$, the projection $\operatorname{Ker} \partial \to H(C, \partial)$ induces an isomorphism of $\operatorname{Ker} S \to H(C, \partial)$. Similarly, the projection $\operatorname{Ker} d \to H(C, d)$ onto the cohomology group $H(C, d)$ induces an isomorphism of $\operatorname{Ker} S \to H(C, d)$. Thus each equivalence class of $H(C, \partial)$ contains a unique harmonic cycle from $\operatorname{Ker} S$, and each equivalence class of $H(C, d)$ contains a unique harmonic cocycle from $\operatorname{Ker} S$.

Since the operators b and ∂ are also disjoint, the laplacian operator $L = b\partial + \partial b$ induces a second Hodge decomposition of C: $C = \operatorname{Ker} L + \operatorname{Im} L$. Identical statements to those made about the first decomposition also hold for this one. In particular, $\operatorname{Ker} L$ plays a similar topological role to that played by $\operatorname{Ker} S$. Since $H(C, \partial)$ is isomorphic to both $\operatorname{Ker} S$ and $\operatorname{Ker} L$, we have described isomorphisms $\psi_{S,L}$: $\operatorname{Ker} L \to \operatorname{Ker} S$ and $\psi_{L,S}$: $\operatorname{Ker} S \to \operatorname{Ker} L$.

2.5. The operator $\psi_{S,L}$, which turns out to be of prime importance, has been determined in terms of other operators which can be described explicitly. L can be inverted on $\operatorname{Im} L$. Defining $L_0 \in \operatorname{End} C$ to be the inverse of L on $\operatorname{Im} L$ and 0 on $\operatorname{Ker} L$, we have $LL_0 = L_0 L = I$ on $\operatorname{Im} L$, $= 0$ on $\operatorname{Ker} L$.

Now let $\pi : \mathfrak{r} \to \operatorname{End} \wedge \mathfrak{r}$ be the adjoint representation. In terms of π define the operator $E \in \operatorname{End} C$ by $E = 2 \sum_{\varphi \in \Delta_+} \pi(e_\varphi)\pi(e_{-\varphi})$. It has been shown that E is related to S and L by the formula $S = L + E$. Finally, define $R \in \operatorname{End} C$ by $R = -L_0 E$. R is clearly nilpotent so that $(I - R)^{-1} = I + R + R^2 + \cdots + R^N$ where $R^{N+1} = 0$. It has been shown that the operator $\psi_{S,L} = (I - R)^{-1}$.

2.6. We will denote by W the Weyl group of $\mathfrak{g}$. For any $\sigma \in W$ we define a subset of $\Delta_+ : \Phi_\sigma = (\sigma\Delta_-) \cap \Delta_+$. The expression of σ in terms of reflections is not unique, but the set Φ_σ is unique. We will consider the elements of W as expressed by means of these sets.

Corresponding to a particular parabolic subalgebra $\mathfrak{u}$, there is a subset W^1 of W defined by $W^1 = \{\sigma \in W \mid \Phi_\sigma \subset \Delta(\mathfrak{n})\}$. When $\mathfrak{u} = \mathfrak{b}$, then $\mathfrak{n} = \mathfrak{m}$ and $W^1 = W$. For each $\sigma \in W$ denote by $n(\sigma)$ the number of roots in Φ_σ. Then for any integer j define $W^1(j) = \{\sigma \in W^1 \mid n(\sigma) = j\}$. For example, $\sigma \in W(1)$ precisely when Φ_σ is a simple root and σ is the corresponding simple reflection. Let $w^1(j)$ be the number of elements in $W^1(j)$.

Let G be a simply-connected Lie group whose Lie algebra is $\mathfrak{g}$. Let $U \subset G$ be the connected Lie subgroup corresponding to $\mathfrak{u} \subset \mathfrak{g}$. Consider the compact homogeneous space G/U. We are interested in the singular cohomology of G/U with complex coefficients. It is zero in odd dimensions, and in dimension $2j$ is a complex vector space of dimension $w^1(j)$. A basis can be given for the cohomology group of G/U whose elements, the generalized Schubert classes, are dual to the homology classes of the generalized Schubert cells in G/U. There is a basis element corresponding to each $\sigma \in W^1$.

The cohomology ring of G/U with complex coefficients is isomorphic as a ring to $H(C, d)$, the product of the latter being induced by exterior multiplication. For any $\sigma \in W^1$ we consider the equivalence class in $H(C, d)$ which passes under this isomorphism into the basis element indexed by σ, and we choose from this class its unique harmonic representative $s_1^\sigma = s^\sigma / \lambda^\sigma \in \operatorname{Ker} S$. These harmonic elements have been fully described. λ^σ is a normalization coefficient. $s^\sigma \in \operatorname{Ker} S$ is defined by first defining an element $h^\sigma \in \operatorname{Ker} L$ and then putting $s^\sigma = \psi_{S,L} h^\sigma$. In § 3, where the discussion of our own results begins, the definition of h^σ will be recalled. Only in § 5 will the properly normalized Schubert classes s_1^σ be used. For the problem investigated here, the s^σ's are the natural elements.

3. Preliminary discussion

3.1. We have $H(C, d) = \sum_{\sigma \in W^1} C\{s_1^\sigma\}$. The problem we have studied is that of multiplication in this ring in terms of the basis of Schubert classes. Actually we have worked with the unnormailzed classes s^σ. Specifically, let $\sigma \in W^1(k)$ and $\xi \in W^1(l)$. Then $\{s^\sigma\} \cdot \{s^\xi\} = \sum_{\eta \in W^1(k+l)} c_{\sigma\xi}^\eta \{s^\eta\}$. We will develop a method for determining the coefficients $c_{\sigma\xi}^\eta$. This problem is equivalent to the following one. s^σ and $s^\xi \in \operatorname{Ker} S$; so they are both cycles and cocycles. Because d is a derivation, $s^\sigma \wedge s^\xi$ is a cocycle, but it is rarely a cycle. However, one can choose $x_{\sigma\xi} \in C^{2k+2l-1}$ such that $s^\sigma \wedge s^\xi + dx_{\sigma\xi} \in \operatorname{Ker} S$ and is thus both a cycle and a cocycle. Although $x_{\sigma\xi}$ is not unique, $dx_{\sigma\xi}$ is unique. So the coefficients $c^\eta_{\sigma\xi}$ also appear in $s^\sigma \wedge s^\xi + dx_{\sigma\xi} = \sum_{\eta \in W^1(k+l)} c^\eta_{\sigma\xi} s^\eta$.

3.2. We wish to compare the problem for a general parabolic $\mathfrak{u} \supset \mathfrak{b}$ with that for $\mathfrak{b}$ itself. $H(C, d)$ in the case of general $\mathfrak{u}$ can be regarded as a subspace of $H(C, d)$ in the case of $\mathfrak{b}$, namely, the subspace spanned by the Schubert classes indexed by $W^1 \subset W$. We will show that the ring structure is not affected by regarding it in this way. For this purpose we temporarily introduce the superscripts U and B to distinguish the two cases. In the case of $\mathfrak{b}$ the ring involves some products which have no meaning in the case of $\mathfrak{u}$, but they do not concern us now. Let $\sigma, \xi \in W^1$. Regarding them as elements of W and looking at the cohomology ring of G/B, we would write:

$$s^{\sigma B} \wedge s^{\xi B} + d^B x^B_{\sigma\xi} = \sum_{\eta \in W(k+l)} c^{\eta B}_{\sigma\xi} s^{\eta B} . \tag{3.2.1}$$

On the other hand, regarding them as elements of W^1 and looking at the cohomology ring of G/U, we would write the same equation with B replaced by U.

Proposition 3.2. *$c^{\eta B}_{\sigma\xi} = c^{\eta U}_{\sigma\xi}$ for $\eta \in W^1$, $= 0$ for $\eta \notin W^1$.*

Proof. It has been shown that the elements s^σ are invariant in the sense that $s^{\sigma U} = s^{\sigma B}$. Because $\mathfrak{n} \subset \mathfrak{m}$, $\wedge \mathfrak{r}^U \subset \wedge \mathfrak{r}^B$. Because $\mathfrak{h} \subset \mathfrak{g}_1$, $C^U \subset C^B$. From the way that ∂ and d are defined, it is obvious that ∂^U and d^U are ∂^B and d^B restricted to C^U. So $0 = \partial^U\{s^{\sigma U} \wedge s^{\xi U} + d^U x^U_{\sigma\xi}\} = \partial^B\{s^{\sigma B} \wedge s^{\xi B} + d^B x^U_{\sigma\xi}\}$ identically. This implies that, having chosen $x^U_{\sigma\xi} \in C^U \subset C^B$, one may chose $x^B_{\sigma\xi} = x^U_{\sigma\xi}$. Then the lefthand sides of (3.2.1) and the corresponding equation with B replaced by U are equal. Upon equating the righthand sides and replacing $s^{\eta U}$ by $s^{\eta B}$ we have $\sum_{\eta \in W(k+l)} c^{\eta B}_{\sigma\xi} s^{\eta B} = \sum_{\eta \in W^1(k+l)} c^{\eta U}_{\sigma\xi} s^{\eta B}$. Since the $\{s^{\eta B} \mid \eta \in W(k + l)\}$ are a basis of $(\operatorname{Ker} S^B)^{2k+2l}$, the assertion follows. q.e.d.

Implicit in this proposition is the first information we have about vanishing products.

Corollary 3.2. *If $\sigma, \xi \in W$ occur together in W^1 corresponding to some $\mathfrak{u} \supset \mathfrak{b}$ and if $\eta \notin W^1$, then $c^\eta_{\sigma\xi} = 0$.*

The import of this proposition is that the multiplication table for any G/U is contained in the single table for G/B. So to solve this single problem is to solve all the others as well. This is not quite as ideal as it sounds since the single problem is the most difficult of all. However, because the elements $h^{\sigma B}$ are far more tractable than the $h^{\sigma U}$, we have of necessity limited our attention to the all-encompassing case $\mathfrak{u} = \mathfrak{b}$. In this case, $\mathfrak{n} = \mathfrak{m}$, $\mathfrak{g}_1 = \mathfrak{h}$, and $\mathfrak{r} = \mathfrak{m} + \mathfrak{m}^*$, and it is this case alone that is discussed in the following sections.

3.3. We now recall the description of the Schubert classes in some detail because the procedure we will follow in multiplying them is patterned on that one would follow in doing a specific example. We will use the following notations. If $\Phi \subset \Delta$ is any subset of roots, specifically $\Phi = \{\varphi_1, \varphi_2, \cdots, \varphi_n\}$, then we will denote $e_\Phi = e_{\varphi_1} \wedge e_{\varphi_2} \wedge \cdots \wedge e_{\varphi_n}$. In this case, when $-\Phi = \{-\varphi_1, -\varphi_2, \cdots, -\varphi_n\}$, $e_{-\Phi} = e_{-\varphi_n} \wedge e_{-\varphi_{n-1}} \wedge \cdots \wedge e_{-\varphi_1}$. We will denote $\langle\Phi\rangle = \varphi_1 + \varphi_2 + \cdots + \varphi_n$.

Now consider $\mathfrak{r}$. We know $\{e_\varphi \mid \varphi \in \Delta\}$ is a basis of $\mathfrak{r}$. So for $n = 1, 2, \cdots$ the elements e_Φ for each distinct subset of Δ containing n roots form a basis of $\wedge \mathfrak{r}^n$. Because $[x, e_\varphi] = \langle\varphi, x\rangle e_\varphi$ for $x \in \mathfrak{h}$, the adjoint action of x on $e_\Phi \in \wedge \mathfrak{r}^n$ is multiplication by the scalar $\langle\langle\Phi\rangle, x\rangle$. Then it is clear that C is the subspace of $\wedge \mathfrak{r}$ spanned by 1 and the elements of the form e_Φ for which $\langle\Phi\rangle = 0$ $(n = 2, 3, \cdots)$. We shall use them as a basis for C. It is also clear that C can be decomposed into a direct sum $C = \sum_{\xi \in \Lambda} C_\xi$ where $\Lambda \subset \mathfrak{h}'$ and $C_\xi = (\wedge \mathfrak{m})^\xi \wedge (\wedge \mathfrak{m}^*)^{-\xi}$. $(\wedge \mathfrak{m})^\xi$ is the subspace of $\wedge \mathfrak{m}$ on which the adjoint action of $x \in \mathfrak{h}$ is multiplication by $\langle\xi, x\rangle$. Since any basis element of C can be written as $e_\Phi \wedge e_{-\Psi}$ where $\Phi, \Psi \subset \Delta_+$ and $\langle\Phi\rangle = \langle\Psi\rangle$, this element clearly lies in C_ξ with $\xi = \langle\Phi\rangle$.

Put $g = \frac{1}{2}\langle\Delta_+\rangle$ and $\xi_\sigma = g - \sigma g$ for any $\sigma \in W$. Cleary $\xi_\sigma = \langle\Phi_\sigma\rangle$. For any $\xi \in \mathfrak{h}'$ define the scalar $b(\xi) = |g|^2 - |g - \xi|^2$ where $|\xi|^2 = (\xi, \xi)$. It has been shown that on the subspace C_ξ the action of the operator L is simply multiplication by the scalar $b(\xi)$. We also know that $\xi_\sigma \in \Lambda$ and that $b(\xi) = 0$ for $\xi \in \Lambda$ if and only if $\xi = \xi_\sigma$ for some $\sigma \in W$. In general the spaces C_ξ are not 1-dimensional, but the spaces C_{ξ_σ} are 1-dimensional.

Let $\sigma \in W(n)$; then $\xi_\sigma = \langle\Phi_\sigma\rangle$. The element h^σ mentioned in § 2 can now be defined. $h^\sigma = (2\pi i)^{-n} e_{\Phi_\sigma} \wedge e_{-\Phi_\sigma}$. Clearly $h^\sigma \in C_{\xi_\sigma} \subset \operatorname{Ker} L$. Since $b(\xi) \neq 0$ for $\xi \neq \xi_\sigma$, the elements h^σ form a basis of $\operatorname{Ker} L$. We shall refer to the elements of this basis as terms of the form h^σ.

Recall now that $C = \operatorname{Ker} L + \operatorname{Im} L$. We have chosen a basis for C and have noted that the terms of the form h^σ are a basis of $\operatorname{Ker} L$. We now show that the remaining basis elements form a basis of $\operatorname{Im} L$.

Lemma 3.3. *Let $\Phi, \Psi \subset \Delta_+$, and $\xi = \langle\Phi\rangle = \langle\Psi\rangle \neq \xi_\sigma$ for any $\sigma \in W$. For all such choices, the elements $e_\Phi \wedge e_{-\Psi}$ form a basis of* $\operatorname{Im} L$.

Proof. $e_\Phi \wedge e_{-\Psi} \in C_\xi$. But $\xi \neq \xi_\sigma$ for any $\sigma \in W$ implies $b(\xi) \neq 0$. Thus $(e_\Phi \wedge e_{-\Psi})/b(\xi) \in C_\xi$, and $L((e_\Phi \wedge e_{-\Psi})/b(\xi)) = e_\Phi \wedge e_{-\Psi}$. So $e_\Phi \wedge e_{-\Psi} \in \operatorname{Im} L$. Each of these basis elements of C is in $\operatorname{Im} L$. Since the rest are in $\operatorname{Ker} L$, these form a basis of $\operatorname{Im} L$.

Remark 3.3. We have proved that when an element of $\operatorname{Im} L$ is written in terms of our basis for C, the expression contains no terms of the form h^σ.

3.4. $h^\sigma \in \operatorname{Ker} L \subset \operatorname{Ker} \partial$ for any $\sigma \in W$ implies that $\{h^\sigma\}$ is a well-defined homology class in $H(C, \partial)$. However, in general $dh^\sigma \neq 0$. As described in § 2, h^σ has a unique harmonic representative $s^\sigma = \psi_{S,L} h^\sigma \in \operatorname{Ker} S$. And, since $\psi_{S,L}$ is known explicitly, s^σ can be computed:

$$s^\sigma = (I - R)^{-1} h^\sigma = (2\pi i)^{-n} \sum_{j=0}^{N} R^j e_{\Phi_\sigma} \wedge e_{-\Phi_\sigma} .$$

In the case that $\sigma \in W(1)$ and thus $\Phi_\sigma = \{\alpha\}$ where α is a simple root, a much simpler expression for s^σ has been given. Then $s^\sigma = (2\pi i)^{-1} \sum_{\varphi \in \Delta_+} n_\alpha(\varphi) \cdot e_\varphi \wedge e_{-\varphi}$ where $n_\alpha(\varphi)$ is the integral coefficient of α in the expression for φ in terms of the simple roots.

With respect to the basis for C, h^σ consists of a single term whereas, in general, s^σ consists of many terms. Even in simple cases the elements s^σ can be very complicated. However, they have a property which, in certain situations, will enable us to ignore their complexity. In the formula for s^σ above, the leading term ($j = 0$) is h^σ. In fact, we have

Proposition 3.4. *h^σ is the only term of the form $h^\nu, \nu \in W$, which appears in s^σ.*

Proof. There is a unique boundary ∂y_σ such that $s^\sigma = h^\sigma + \partial y_\sigma$. $\partial y_\sigma \in \operatorname{Im} \partial \subset \operatorname{Im} L$. So by Remark 3.3 no terms of the form $h^\nu, \nu \in W$, will occur in ∂y_σ. So h^σ itself is the only such term in s^σ.

Remark 3.4. Because $\{h^\sigma \mid \sigma \in W\}$ are a basis of $\operatorname{Ker} L$, $\{s^\sigma \mid \sigma \in W\}$ are a basis of $\operatorname{Ker} S$. In the problem which concerns us, we will have $z \in \operatorname{Ker} S$ written in terms of the basis of C and will wish instead to know it in the form $z = \sum_{\sigma \in W} b^\sigma s^\sigma$. Approached directly, this would have been difficult for two reasons. First, unless $\sigma \in W(1)$, we may not know the proper expression for s^σ. Second, even if we did know the s^σ's, we would have trouble recognizing them in z because a single basis element of C can appear in many different s^σ's. As a result of Proposition 3.4, neither of these difficulties need concern us. Let $z \in \operatorname{Ker} S$. Like any element of C, z can be written as $z = \sum_{\sigma \in W} \bar{b}^\sigma h^\sigma + w = \sum_{\sigma \in W} b^\sigma s^\sigma$ where $w \in \operatorname{Im} L$. By Remark 3.3, w contains no term of the form h^ν. By Proposition 3.4, s^σ contains h^σ and no other term of the form h^ν. So necessarily $\bar{b}^\sigma = b^\sigma$. Thus we can determine the representation of $z \in \operatorname{Ker} S$ in terms of the basis $\{s^\sigma \mid \sigma \in W\}$ by simply observing the terms of the form h^σ.

4. Computation of products

4.1. We now return to the problem of determining the unique $dx_{\sigma\xi}$ for which $s^\sigma \wedge s^\xi + dx_{\sigma\xi} \in \operatorname{Ker} S$. For this purpose we continue the discussion of the laplacians L and S which was begun in § 2. S can be inverted on $\operatorname{Im} S$. Defining S_0 to be the inverse of S on $\operatorname{Im} S$ and 0 on $\operatorname{Ker} S$, we have $SS_0 = S_0 S = I$ on $\operatorname{Im} S$, $= 0$ on $\operatorname{Ker} S$.

It is clear from the definition of S and L that $dS = Sd$, $\partial S = S\partial$, and $\partial L = L\partial$. Also we have

Lemma 4.1. *$dS_0 = S_0 d$, $\partial S_0 = S_0 \partial$, and $\partial L_0 = L_0 \partial$.*

Proof. $dS_0 = S_0 d$ on $\operatorname{Ker} S$ since both S_0 and d vanish there. So to prove this formula for C it suffices to show that it also holds for $\operatorname{Im} S$. Now $\operatorname{Im} S$ is closed under S, S_0, and d (also ∂). Since $S_0 S = SS_0 = I$ there, $dS_0 = S_0 S d S_0 = S_0 d S S_0 = S_0 d$. The other two assertions follow in the same manner. q.e.d.

Now consider the subspace $\operatorname{Im} \partial \subset \operatorname{Im} L \cap \operatorname{Im} S$. Clearly $\operatorname{Im} \partial$ is closed under S. Since S is an automorphism of $\operatorname{Im} S$, S is also an automorphism of $\operatorname{Im}\partial$. Similarly L is an automorphism of $\operatorname{Im}\partial$. Thus S_0 and L_0 are also automorphisms of $\operatorname{Im}\partial$.

We can now express the action of S_0 on $\operatorname{Im}\partial$ in terms of operators which are known explicitly.

Proposition 4.1. *On* $\operatorname{Im}\partial$, $S_0 = (I - R)^{-1}L_0$.

Proof. Because $S = L + E$, $\operatorname{Im}\partial$ is also closed under E and thus under $R = -L_0E$. Since $\operatorname{Im}\partial \subset \operatorname{Im} L \cap \operatorname{Im} S$, $LL_0 = L_0L = SS_0 = S_0S = I$ on $\operatorname{Im}\partial$. Then $L_0S = L_0L + L_0E = I - R$. So $S_0 = S_0LL_0 = (L_0S)^{-1}L_0 = (I - R)^{-1}L_0$ on $\operatorname{Im}\partial$. q.e.d.

Note that this formula is not a proper expression for S_0 on all of $\operatorname{Im} S$.

4.2. We can now give an explicit formula relating $dx_{\sigma\xi}$ to the Schubert classes s^σ and s^ξ.

Theorem 4.2. *If* $s^\sigma \wedge s^\xi + dx_{\sigma\xi} \in \operatorname{Ker} S$, *then* $dx_{\sigma\xi} = -d\partial(I-R)^{-1}L_0(s^\sigma \wedge s^\xi)$.

Proof. Because $s^\sigma \wedge s^\xi + dx_{\sigma\xi} \in \operatorname{Ker} S$, $Sdx_{\sigma\xi} = -S(s^\sigma \wedge s^\xi)$; so $S_0Sdx_{\sigma\xi} = -S_0S(s^\sigma \wedge s^\xi)$. Because $dx_{\sigma\xi} \in \operatorname{Im} d \subset \operatorname{Im} S$, $S_0Sdx_{\sigma\xi} = dx_{\sigma\xi}$. Moreover, $d(s^\sigma \wedge s^\xi) = 0$ implies that $S(s^\sigma \wedge s^\xi) = d\partial(s^\sigma \wedge s^\xi)$. So we have shown that $dx_{\sigma\xi} = -S_0d\partial(s^\sigma \wedge s^\xi) = -dS_0\partial(s^\sigma \wedge s^\xi)$. Then, applying Proposition 4.1, $dx_{\sigma\xi} = -d(I - R)^{-1}L_0\partial(s^\sigma \wedge s^\xi)$. ∂ commutes with L_0; because it commutes with L and S, it also commutes with E, R, and $(I - R)^{-1}$. So $dx_{\sigma\xi} = -d\partial(I - R)^{-1}L_0(s^\sigma \wedge s^\xi)$.

Corollary 4.2. *Given the cocycle* $s^\sigma \wedge s^\xi$, $[I - d\partial(I - R)^{-1}L_0](s^\sigma \wedge s^\xi) \in \operatorname{Ker} S$ *is the unique harmonic cocycle which represents the same cohomology class.*

4.3. According to Remark 3.4, we can write this harmonic cocycle in terms of the basis of Schubert classes by simply observing the terms of the form h^σ. For this reason, we can replace it by another expression which is not even in $\operatorname{Ker} S$ but in which the coefficients of the terms of the form h^σ remain the same. In doing this we will be able to replace the operator $[I - d\partial(I - R)^{-1}L_0]$ by a simpler one.

Similar to R, we define an operator $\bar{R} = -EL_0$. Like R, $\bar{R}$ is nilpotent; $R^{N+1} = 0$ implies $\bar{R}^{N+2} = 0$.

Define the projection operator P by $P = I$ on $\operatorname{Ker} L$ and $P = 0$ on $\operatorname{Im} L$. Then the effect of the composite $\psi_{S,L}P$ on any element of C is first to annihilate all terms but those of the form h^σ and second to replace them by the corresponding s^σ's.

Theorem 4.3. *In terms of the Schubert classes, the product* $\{s^\sigma\}\cdot\{s^\xi\}$ *is* $\{\psi_{S,L}P(I - \bar{R})^{-1}(s^\sigma \wedge s^\xi)\}$.

Proof. From Theorem 4.2 we have

$$\begin{aligned} dx_{\sigma\xi} &= -d\partial(I - R)^{-1}L_0(s^\sigma \wedge s^\xi) \\ &= [-S(I - R)^{-1}L_0 + \partial d(I - R)^{-1}L_0](s^\sigma \wedge s^\xi) \end{aligned}$$

$$= [-E(I-R)^{-1}L_0 - L(I-R)^{-1}L_0 + \partial d(I-R)^{-1}L_0](s^\sigma \wedge s^\xi)$$
$$= -E(I-R)^{-1}L_0(s^\sigma \wedge s^\xi) + z \qquad \text{where } z \in \operatorname{Im} L.$$

One computes that $-E(I-R)^{-1}L_0 = [(I-\bar{R})^{-1} - I]$. The harmonic cocycle which represents $\{s^\sigma\}\cdot\{s^\xi\} = \{s^\sigma \wedge s^\xi\}$ is $s^\sigma \wedge s^\xi + dx_{\sigma\xi} = (I-\bar{R})^{-1}(s^\sigma \wedge s^\xi) + z$. Because $Pz = 0$, when this is written in terms of the Schubert classes, it becomes $\psi_{S,L}P(I-\bar{R})^{-1}(s^\sigma \wedge s^\xi)$.

Remark 4.3. Although Theorems 4.2 and 4.3 are stated and proved for the product of two Schubert classes, it is clear that the entire discussion applies equally well to the product of any number of classes.

4.4. Although Theorem 4.3 can, in principle, be applied to determine the product of any Schubert classes, it is sufficiently complicated that general results are difficult to obtain from it. The difficulty is due to the operators $(I-R)^{-1}$ and $(I-\bar{R})^{-1}$. Ordinarily one begins with a knowledge of Φ_σ and thus h^σ. The first of these operators is necessary in determining $s^\sigma = (I-R)^{-1}h^\sigma$. Then the exterior product of two or more such elements is formed and the operator $(I-\bar{R})^{-1}$ is applied.

These two operators can be dispensed with in certain special cases. If we consider a product of k Schubert classes, each corresponding to a $\sigma \in W(1)$, then we may use the simple formula for s^σ, thus dispensing with the operator $(I-R)^{-1}$. On the other hand, if we are concerned with a $\mathfrak{u}$ such that $\mathfrak{n}$ is abelian, as it is whenever G/U is a complex symmetric space, then $\partial^U = 0$. If W^1 corresponds to this $\mathfrak{u}$ and we choose $\sigma_i \in W^1$, $1 \leq i \leq k$, then the product $s^{\sigma_1} \wedge \cdots \wedge s^{\sigma_k} \in \operatorname{Ker} \partial \cap \operatorname{Ker} d = \operatorname{Ker} S$. In this case it is not necessary to use Theorem 4.3. It is clear that $\{s^{\sigma_1}\} \bullet \cdots \bullet \{s^{\sigma_k}\} = \{\psi_{S,L}P(s^{\sigma_1} \wedge \cdots \wedge s^{\sigma_k})\}$. Most of the results described in the next section apply to cases where one or both of these simplifications occur.

5. Various applications

5.1. In Corollary 3.2, we have already seen a condition which implies that the coefficient $c^\eta_{\sigma\xi} = 0$. We can now describe further circumstances in which the same null result occurs. Note that the given condition will be sufficient but not necessary.

Proposition 5.1. *Let $\sigma, \xi, \eta \in W$. If there is a simple root α such that there is $\varphi \in \Phi_\sigma \cup \Phi_\xi$ for which $n_\alpha(\varphi) > 0$ but there is no $\psi \in \Phi_\eta$ for which $n_\alpha(\psi) > 0$, then $c^\eta_{\sigma\xi} = 0$.*

Proof. Suppose there is $\varphi \in \Phi_\sigma \cup \Phi_\xi$ and $n_\alpha(\varphi) > 0$. Then e_φ is a factor of either h^σ or h^ξ. By the nature of the operator R, either every term of s^σ or every term of s^ξ will contain a factor e_θ for which $n_\alpha(\theta) > 0$. So the same can be said for $s^\sigma \wedge s^\xi$. By the nature of the operator $\bar{R}$, every term of $(I-\bar{R})^{-1}(s^\sigma \wedge s^\xi)$ will contain a factor e_ψ for which $n_\alpha(\psi) > 0$. So, by Theorem 4.3, if it were true that $c^\eta_{\sigma\xi} \neq 0$, then h^η would contain a factor e_ψ

for which $n_\alpha(\psi) > 0$; and it would follow that there is $\psi \in \Phi_\tau$ for which $n_\alpha(\psi) > 0$, contrary to hypothesis.

5.2. In the remainder of this section, we will present several formulas involving products of Schubert classes and will phrase them in terms of the true classes, the $s_1^\sigma = s^\sigma/\lambda^\sigma$. The normalization coefficient λ^σ had formerly been computed only in the case $\sigma \in W(1)$ and thus $\Phi_\sigma = \{\alpha\}$ where α is a simple root. In that case $\lambda^\sigma = 2/(\alpha, \alpha)$. It is now known [2] that in general these coefficients are given by the following formula. Suppose $\sigma \in W(n)$; then its inverse $\sigma^{-1} \in W(n)$. Let $\Phi_{\sigma^{-1}} = \{\varphi_1, \cdots, \varphi_n\}$. Then $1/\lambda^\sigma = (g, \varphi_1)\cdots(g, \varphi_n)$.

The preceding discussion applies in its entirety to the s_1^σ's as well as to the s^σ's. In computing a product, one has only to replace the coefficients $c_{\sigma\xi}^\eta$ by $[\lambda^\eta/(\lambda^\sigma\lambda^\xi)]c_{\sigma\xi}^\eta$. One sees from their definition that the coefficients λ^σ have the same invariance property that the s^σ's have; so the s_1^σ's have this same property, and Proposition 3.2 and Corollary 3.2 hold for them as well. Clearly Proposition 5.1 is also unaltered.

5.3. The simplest nontrivial product is $\{s_1^\sigma\} \cdot \{s_1^\xi\}$ where $\sigma, \xi \in W(1)$. In this case we can easily write down answers which apply to all cases. Let $\sigma, \xi \in W(1)$. Let α and β be the simple roots such that $\Phi_\sigma = \{\alpha\}$ and $\Phi_\xi = \{\beta\}$. Furthermore, if γ is any simple root, let $\eta(\gamma)$ be that element in $W(1)$ for which $\Phi_{\eta(\gamma)} = \{\gamma\}$. We will denote the simple roots by Π, the product in W by $\circ$, and the Cartan integers by $c_{\alpha\beta} = 2(\beta, \alpha)/(\beta, \beta)$. Then the following formulas hold.

Theorem 5.3.
$$\{s_1^\sigma\} \cdot \{s_1^\xi\} = \{s_1^{\sigma\circ\xi}\} \qquad \text{if } c_{\alpha\beta} = 0\,,$$
$$\{s_1^\sigma\} \cdot \{s_1^\xi\} = \{s_1^{\sigma\circ\xi}\} + \{s_1^{\xi\circ\sigma}\} \qquad \text{if } c_{\alpha\beta} < 0\,,$$
$$\{s_1^\sigma\} \cdot \{s_1^\sigma\} = -\sum_{\gamma\in\Pi} c_{\alpha\gamma}\{s_1^{\sigma\circ\eta(\gamma)}\}\,.$$

Proof. The details of the computation will not be given here. They are considerably simplified by facts about the structure of the sets Φ_σ which have not been mentioned here. Because $\sigma, \xi \in W(1)$, the simple formulas for s^σ and s^ξ can be used. The first of these statements is almost immediate. Since $\Phi_\sigma = \{\alpha\}$ and $\Phi_\xi = \{\beta\}$, we know from Proposition 5.1 that the only $\eta \in W(2)$ for which $c_{\sigma\xi}^\eta$ could be nonzero would be one for which α and β both figure in Φ_η. Because $c_{\alpha\beta} = 0$, there is only one such $\eta \in W(2)$, and it is $\eta = \sigma\circ\xi = \xi\circ\sigma$ for which $\Phi_{\sigma\circ\xi} = \{\alpha, \beta\}$. It is clear that $h^{\sigma\circ\xi}$ is the leading term of $s^\sigma \wedge s^\xi$, and also that this term will never occur in $\bar{R}^j(s^\sigma \wedge s^\xi)$ for $j \geq 1$. So the result is obtained by simply observing the coefficient of the leading term. The two other formulas are established in a similar way except that $\bar{R}$ comes into play. One observes first, with the help of Proposition 5.1, the small number of η's in $W(2)$ for which $c_{\sigma\xi}^\eta$ could be nonzero. Then one observes in each case the small number of ways in which h^η can occur in $(I - \bar{R})^{-1}(s^\sigma \wedge s^\xi)$. Finally one computes the coefficients of these terms alone and combines them. The highest power of $\bar{R}$ which enters is $\bar{R}^3$ and that occurs only in establishing the second formula in the case $c_{\alpha\beta} = -3$.

5.4. Let α be any simple root. Put $A(\alpha) = \{\varphi \in \Delta_+ \mid n_\alpha(\varphi) > 0\}$ and $N(\alpha) = \text{Max}\,\{n_\alpha(\varphi) \mid \varphi \in A(\alpha)\}$. We will consider now the case where $N(\alpha) = 1$.

Theorem 5.4. *Let $\sigma \in W(1)$ and $\Phi_\sigma = \{\alpha\}$ where $N(\alpha) = 1$. In the product $\{s_1^\sigma\}^k$, the coefficient of $\{s_1^\eta\}$, $\eta \in W(k)$, is $(\lambda^\eta/(\lambda^\sigma)^k)k!$ if $\Phi_\eta \subset A(\alpha)$ and 0 otherwise.*

Proof. One can see that the span of the $\{e_\varphi \mid \varphi \in A(\alpha)\}$ is a Lie subalgebra $\mathfrak{n}$ corresponding to a parabolic $\mathfrak{u}$. Because $N(\alpha) = 1$, $\mathfrak{n}$ is abelian. Our $\sigma \in W^1$ corresponding to this $\mathfrak{u}$. So, as mentioned at the end of § 4, the operator $(I - \bar{R})^{-1}$ can be replaced by I. It is then a question of what terms of the from h^η appear in $(s^\sigma)^k$. From the simple formula for s^σ, $\sigma \in W(1)$, we see they correspond exactly to those $\eta \in W(k)$ for which $\Phi_\eta \subset A(\alpha)$. In that case the term of the form h^η occurs with coefficient 1, $k!$ times.

5.5. In the case where $N(\alpha) > 1$, the problem is much more difficult; but an answer can be given in the special case of the highest nonzero power. Since $A(\alpha)$ and its complement in Δ_+ are both closed under addition of roots, there is a $\xi(\alpha) \in W$ for which $\Phi_{\xi(\alpha)} = A(\alpha)$. Denote by $D(\alpha)$ the number of roots in the set $A(\alpha)$ and by $H(\alpha)$ the product $H(\alpha) = \prod_{\varphi \in A(\alpha)} n_\alpha(\varphi)$.

Theorem 5.5. *Let $\sigma \in W(1)$ and $\Phi_\sigma = \{\alpha\}$. Then*

$$\{s_1^\sigma\}^{D(\alpha)} = (\lambda^{\xi(\alpha)}/(\lambda^\sigma)^{D(\alpha)})D(\alpha)!\ H(\alpha)\{s_1^{\xi(\alpha)}\}\ .$$

Proof. Clearly $(s_1^\sigma)^{D(\alpha)} = (1/(\lambda^\sigma)^{D(\alpha)})D(\alpha)!\ H(\alpha)h^{\xi(\alpha)}$. But $\bar{R}h^{\xi(\alpha)} = 0$. Hence the result follows upon introduction of $\lambda^{\xi(\alpha)}$.

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